Properties of Splines in Tension

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Rigorous arguments are given establishing convergence rates and asymptotic behavior of interpolatory tension splines with variable tension. It is shown that for sufficiently smooth data, convergence is $O(h^4)$ for uniformly bounded tension parameters. For large tension parameters the tension spline is essentially locally linear and applications of this fact are given which allow one to construct convex or monotone approximants.

1. INTRODUCTION

Tension splines were first introduced by Schweikert [8] as a means of eliminating extraneous inflection points in curve fitting by cubic splines. Later it was recognized that a tension spline is an L-spline corresponding to the differential operator $L = D^2 - \rho D$ where $D = d/dx$ and ρ, the tension parameter, is to be chosen. Spath [9] modified the tension spline so that different values of the tension parameter could be chosen in different regions of the domain. This is an example of a piecewise L-spline in the sense of Prenter [6]. In particular, given a partition $\{a = x_1 < x_2 < \cdots < x_{N+1} = b\}$ of an interval $[a, b]$ and a set of tension parameters $\{\rho_i\}_{i=1}^N \tau(x)$ is a tension spline if it satisfies

$$\tau \in C^2[a, b] \tag{1.1}$$

$$(D^4 - \rho_i^2 D^2)\tau = 0 \quad \text{in each } (x_i, x_{i+1}). \tag{1.2}$$

The case when all tension parameters are equal is referred to as the case of uniform tension. Note that for a uniform tension of zero τ is a cubic spline, moreover, τ satisfies $(1/\rho^2 D^4 - D^2)\tau = 0$ in each subinterval, so for large ρ it appears that τ tends to a linear spline. Hence the name tension spline: In the interpolatory case when $\tau(x_i) = f_i$ is specified, graphically it appears as if the curve through the data points is being pulled tighter as ρ increases. Cline [1] has written some computer programs and described applications for the tension splines.

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This paper is concerned with questions of convergence and analysis of the behavior for large ρ of interpolatory tension splines. The results of Schultz and Varga [7] on L-splines apply to the case of uniform tension and those of Prenter [6] on piecewise L-splines to that of variable tension, but Theorem 1 in Section 2 of this paper is stronger. Hill [5] has also derived a convergence result for tension splines but this too is not as strong as Theorem 1. It is shown that the convergence is uniformly $O(h^4)$ ($h = \max_i(x_{i+1} - x_i)$) for uniformly bounded tension parameters when the function being approximated is sufficiently smooth. The behavior of τ for fixed partitions as ρ is increased is discussed in Section 3. This, too, has been studied by Schweikert, Spath, and Hill but is developed more fully here. It is shown that on the ith interval as $\rho_i \to \infty$ the tension spline does indeed converge to the linear spline in that interval, moreover, if all tension parameters grow in such a way that the ratio of the largest to the smallest is uniformly bounded then τ converges uniformly to the linear spline on $[a, b]$. The behavior of the derivatives is also studied as well as applications to computing convex interpolants and monotone interpolants.

2. CONVERGENCE RESULTS

As mentioned in the previous section there is a close relationship between tension splines and cubic splines. This is exploited to establish convergence rates for the former since information about the latter is well known. Before stating the main result we introduce a convenient notation for the tension spline which leads to an efficient computational algorithm and also provides the means to study convergence.

Given a partition $\{a = x_1 < x_2 < \cdots < x_{N+1} = b\}$ of $[a, b]$ let $h_i = x_{i+1} - x_i$, $\tau_i'' = \tau''(x_i)$, $S_i = \sinh \rho_i h_i$, $C_i = \cosh \rho_i h_i$, $h = \max_i h_i$ and $f_i = f(x_i)$ where f is the function to be interpolated. From (1.2) $\tau(x)$, the interpolatory tension spline, must satisfy in $[x_i, x_{i+1}]$

$$(D^4 - \rho_i^2 D^2)\tau = 0, \qquad \tau(x_i) = f_i, \qquad \tau(x_{i+1}) = f_{i+1},$$

$$\tau''(x_i) = \tau_i'', \qquad \tau''(x_{i+1}) = \tau_{i+1}''.$$

The solution to this boundary value problem is given by

$$\tau(x) = [\tau_i'' \sinh \rho_i(x_{i+1} - x) + \tau_{i+1}'' \sinh \rho_i(x - x_i)]/(\rho_i^2 S_i) \qquad (2.1)$$

$$+ (f_i - \tau_i''/\rho_i^2)(x_{i+1} - x)/h_i + (f_{i+1} - \tau_{i+1}''/\rho_i^2)(x - x_i)/h_i.$$

This formula results in τ interpolatory and τ'' continuous so the final

requirement that τ' be continuous is used to determine $\{\tau_i''\}$. Two end conditions are also needed which can be selected in a variety of ways [7], e.g.,

type I, $\tau'(a) = f'(a)$, $\tau'(b) = f'(b)$;

type II, $\tau''(a) = \tau''(b) = 0$;

type III, $\tau''(a) = f''(a)$, $\tau''(b) = f''(b)$.

Note that the above type II condition is not the usual one for L-splines which says $\tau'' = \rho\tau'$ at the end points, however, the modification is convenient. Continuity of τ' along with type I end conditions yields the following symmetric, tridiagonal linear system for $\{\tau_i''\}$.

$$d_1\tau_1'' + e_1\tau_2'' = (f_2 - f_1)/h_1 - f'(a) \tag{2.2a}$$

$$e_{i-1}\tau_{i-1}'' + (d_{i-1} + d_i)\,\tau_i'' + e_i\tau_{i+1}'' = (f_{i+1} - f_i)/h_i - (f_i - f_{i-1})/h_{i-1}$$

$$i = 2, 3,..., N \tag{2.2b}$$

$$e_N\tau_N'' + d_N\tau_{N+1}'' = f'(b) - (f_{N+1} - f_N)/h_N \tag{2.2c}$$

where

$$e_i = (1/h_i - \rho_i/S_i)/\rho_i^2 \tag{2.3a}$$

$$d_i = (\rho_iC_i/S_i - 1/h_i)/\rho_i^2. \tag{2.3b}$$

For the other two types of end conditions (2.2a) and (2.2c) are trivially modified. When ρ_ih_i is small, loss of significance occurs in (2.3) in which case

$$e_i = h_i(1 - (7/60)\,\rho_i^2h_i^2)/6 \tag{2.4a}$$

$$d_i = h_i(1 - (1/15)\,\rho_i^2h_i^2)/3 \tag{2.4b}$$

should be used (Eq. (2.1) also requires modification in this case). Note that a uniform tension of zero results in (2.4) and (2.2) simplifying to the familiar formula for cubic interpolatory splines. It is easily seen that d_i and e_i are positive and that the coefficient matrix in (2.2) is strictly diagonally dominant. Thus the system can be quickly and easily solved using a special algorithm for tridiagonal, diagonally dominant matrices [2, pp. 118–119]. As is the case for interpolatory cubic splines the matrix is very well conditioned as is shown below.

To establish rates of convergence for τ to f we first compute bounds for $\sigma - \tau$ where σ is the cubic interpolatory spline for the same partition, data and end conditions. Equivalently, σ is the interpolatory tension spline with

uniform tension of zero. Let $\varDelta$ be the vector whose ith component is $\delta_i'' = \sigma_i'' - \tau_i''$, then

$$\delta_1'' + (e_1/d_1)\,\delta_2'' = (h_1/3d_1 - 1)\,\sigma_1'' + (h_1/6 - e_1)/d_1 \cdot \sigma_2'' \qquad (2.5a)$$

$$\frac{e_{i-1}}{(d_{i-1} + d_i)}\,\delta_{i-1}'' + \delta_i'' + \frac{e_i}{(d_{i-1} + d_i)}\,\delta_{i+1}''$$

$$= \frac{h_{i-1}/6 - e_{i-1}}{d_{i-1} + d_i}\,\sigma_{i-1}'' + \left(\frac{h_i + h_{i-1}}{3(d_{i-1} + d_i)} - 1\right)\sigma_i'' + \frac{h_i/6 - e_i}{d_{i-1} + d_i}\,\sigma_{i+1}''$$

$$i = 2, 3,..., N \quad (2.5b)$$

$$(e_N/d_N)\,\delta_N'' + \delta_{N+1}'' = (h_N/6 - e_N)/d_N \cdot \sigma_N'' + (h_N/3d_N - 1)\,\sigma_{N+1}'' . \qquad (2.5c)$$

These equations assume type I end conditions, for the other two types only (2.5b) applies with $\delta_1'' = \delta_{N+1}'' = 0$. Let $\theta_i = \rho_i h_i$ then

$$0 < e_i/d_i = \frac{1}{2}\left(1 + \frac{1}{20}\,\theta_i^2 + \cdots + \frac{6}{(2n + 3)!}\,\theta_i^{2n} + \cdots\right)\Big/$$

$$\left(1 + \frac{1}{10}\,\theta_i^2 + \cdots + \frac{6(n + 1)}{(2n + 3)!}\,\theta_i^{2n} + \cdots\right)$$

$$\leqslant \frac{1}{2} \qquad \text{uniformly in } \theta_i .$$

Similarly

$$0 < (e_{i-1} + e_i)/(d_{i-1} + d_i) \leqslant 1/2,$$
$$0 \leqslant (h_i/6 - e_i)/d_i \leqslant (1/2)\,\theta_i^2,$$
$$0 \leqslant h_i/(3d_i) - 1 \leqslant \theta_i^2,$$

and $0 \leqslant (h_{i-1} + h_i)/[3(d_{i-1} + d_i)] - 1 \leqslant \rho_{max}h^2$, where $\rho_{max} = \max_i \rho_i$. If the linear system (2.5) is written in matrix form as $(I + E)\varDelta = r$ then the above inequalities imply $\| E \|_\infty \leqslant \frac{1}{2}$ and $\| r \|_\infty \leqslant 2\rho_{max}h^2 \max_i | \sigma_i'' |$. Thus

$$\| \varDelta \|_\infty \leqslant \|(I + E)^{-1}\|_\infty \| r \|_\infty \leqslant \| r \|_\infty/(1 - \| E \|_\infty) \leqslant 4\rho_{max}^2 h^2 \max_i | \sigma_i'' | . \qquad (2.6)$$

Here $\| \cdot \|_\infty$ indicates the usual matrix ∞-norm (maximum row sum); in what follows $\| \cdot \|$ is the max–norm for continuous functions. Equation (2.6) provides the crucial step for the main result for this section.

THEOREM 1. *If for a given partition of $[a, b]$ τ is the type I (II, III) inter-polatory tension spline for the data $\{ f(x_i) \}$ corresponding to some choice of*

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tension parameters $\{\rho_i\}$, *and* σ *is the interpolatory cubic spline for the same data and end conditions then*

$$\| D^i(\sigma - \tau)\| \leqslant (26/3)\, \rho_{\max}^2 h^{4-i} \max_j |\, \sigma_j'' \,| \qquad i = 0, 1, 2. \tag{2.7}$$

Proof. Let $\delta = \sigma - \tau$ then $\delta \in C^2[a, b]$ and $\delta(x_i) = \delta(x_{i+1}) = 0$, $i = 1, 2,..., N$. Thus $\exists \xi_i \in (x_i, x_{i+1})$ such that $\delta'(\xi_i) = 0$, and for $x \in (x_i, x_{i+1})$

$$\delta(x) = \int_{x_i}^{x} \delta'(t)\, dt, \qquad \delta'(x) = \int_{\xi_i}^{x} \delta''(t)\, dt. \tag{2.8}$$

But

$$\delta''(x) = [(x - x_i)/h_i - \sinh \rho_i(x - x_i)/S_i]\, \sigma_{i+1}'' + \delta_{i+1}'' \sinh \rho_i(x - x_i)/S_i$$

$$+ [(x_{i+1} - x)/h_i - \sinh \rho_i(x_{i+1} - x)/S_i]\sigma_i'' + \delta_i'' \sinh \rho_i(x_{i+1} - x)/S_i.$$

and from series expansions for the terms in brackets, they can be bounded by $\frac{1}{3}\rho_i^2 h_i^2$. Therefore with (2.6)

$$\| \delta'' \| \leqslant (26/3)\, \rho_{\max}^2 h^2 \max_j |\, \sigma_j'' \,|;$$

(2.7) then follows from (2.8).

This theorem can be combined with various known results for $\| D^i(f - \sigma)\|$ to produce bounds for $\| D^i(f - \tau)\|$ (see e.g., Hall [3, 4]). For example,

COROLLARY 1. *If* $f \in C^4[a, b]$ *and* τ *is the interpolatory tension spline corresponding to type I end conditions then* $\exists K$ *depending on* $\rho_{\max}$, $\| D^2 f\|$, $\| D^4 f\|$ *but independent of* h *such that*

$$\| D^i(f - \tau)\| \leqslant K h^{4-i}, \qquad i = 0, 1, 2.$$

Theorem 1 also provides information on the behavior of τ as the tension parameters tend to zero.

COROLLARY 2. *For a fixed partition, if* $\{\tau^n\}$ *is a sequence of interpolatory tension splines with* $\lim_{n\to\infty} \rho_{\max} = 0$ *then*

$$\lim_{n\to\infty} \| D^i(\tau^n - \sigma)\| = 0 \qquad i = 0, 1, 2,$$

where σ *is the cubic interpolatory spline for the same data and end conditions.*

We conclude this section with a discussion of the conditioning of the coefficient matrix involved in the computation of $\{\tau_i''\}$ (2.2). If this matrix is denoted by A then Gerschgorin's theorem yields

$$\| A \|_2 \leqslant 2 \max_i (C_i - 1)/(\rho_i S_i)$$

and

$$\| A^{-1} \|_2 \leqslant (1/2) \max_i (h_i \rho_i^2 S_i)/(\rho_i h_i C_i + \rho_i h_i - 2S_i)$$

where $\| \cdot \|_2$ for symmetric matrices is the spectral radius. From series expansions it follows that

$$\| A \|_2 \leqslant \max_i h_i = h$$

and

$$\| A^{-1} \|_2 \leqslant \max_i 3/h_i = 3/\min_i h_i$$

and thus the condition number [2, Sect. 8] for this linear system is bounded by $3h/\min_i h_i$ as is true for interpolatory cubic splines.

3. Behavior for Large Values of the Tension Parameter

As mentioned in the introduction tension splines have their main use in mimicking convexity or monotonicity properties of data. If the data corresponds to a function with large second derivatives it may not be possible to construct such an approximation with cubic splines. With tension splines, for a fixed partition, as the tension parameter in one interval is increased the interpolant more or less approaches the linear spline interpolant for that interval and hence reflects the convexity or monotonicity of the data. This result is made precise below and other results are established concerning the behavior of τ_i', τ_i'' for large ρ.

For later use we state inequalities for the terms occurring in the $\{\tau_i''\}$ coefficient matrix. These can be straightforwardly established by elementary calculus.

LEMMA 1. *With d_i and e_i given by (2.3) we have for all $\rho_i > 0$*

$$0 < d_i < 1/\rho_i, \tag{3.1}$$
$$0 < e_i < 1/(4\rho_i), \tag{3.2}$$
$$e_i < 1/(\rho_i^2 h_i), \tag{3.3}$$

and for $\rho_i h_i > 3$

$$1/(2\rho_i) < d_i. \tag{3.4}$$

Two types of tension spline sequences $\{\tau^n\}$ are considered, each for a fixed partition; in one the tension parameters grow locally while in the other all tension parameters grow, i.e.,

Case 1.   For a given k, $2 < k < N$, $\rho_j{}^n$ $j \neq k-1, k, k+1$ is fixed independently of n and $\lim_{n \to \infty} \rho_j{}^n = \infty$ $j = k-1, k, k+1$,

Case 2. $\exists \sigma > 0$ such that for n sufficiently large $\rho_{max}/\rho_{min} \leqslant \sigma$ $(\rho_{min} = \min_i \rho_i{}^n)$ and $\lim_{n \to \infty} \rho_{max} = \infty$ (which implies $\lim_{n \to \infty} \rho_{min} = \infty$).

(Note that superscripts here refer to the index in the sequence not exponents.)

A second preliminary result is needed concerning the asymptotic behavior of τ_i'' for large tension. In order to make the arguments independent of end conditions we let F_j, $1 \leqslant j \leqslant N+1$, be the jth right hand side of the linear system (2.2). For $j = 2, 3, ..., N$

$$F_j = (f_{j+1} - f_j)/h_j - (f_j - f_{j-1})/h_{j-1}. \tag{3.5}$$

For type I end conditions $F_1 = (f_2 - f_1)/h_1 - f'(a)$, for type II $F_1 = 0$, for type III $F_1 = f''(a)$, etc.

LEMMA 2.   *For a fixed partition*

(1)   *when ρ_{max} is sufficiently large (so that $\rho_i h_i > 3 \forall_i$)*

$$\max_i | \tau_i'' | \leqslant 2\rho_{max} \max_j | F_j |, \tag{3.6}$$

(2)   *for a Case 1 sequence of tension splines*

$$\lim_{n \to \infty} (d_{i-1}^n + d_i^n)(\tau^n)_i'' = F_i \qquad i = k, k+1, \tag{3.7}$$

(3)   *for a Case 2 sequence of tension splines*

$$\lim_{n \to \infty} (d_{i-1}^n + d_i^n)(\tau^n)_i'' = F_i \qquad 2 \leqslant i \leqslant N, \tag{3.8}$$

for type 1 end conditions $\lim_{n \to \infty} d_i^n(\tau^n)_i'' = F_i$ $i = 1, N+1$.

Proof.   For simplicity details concerning the end conditions are omitted as well as superscripts for the last two parts. Equation (2.2) implies

$$e_{i-1}\tau_{i-1}''/(d_{i-1} + d_i) + \tau_i'' + e_i\tau_{i+1}''/(d_{i-1} + d_i) = F_i/(d_{i-1} + d_i) \tag{3.9}$$

and by a matrix argument similar to that used in the proof of Theorem 1

$$\max_i | \tau_i'' | \leqslant 2 \max_i | F_i/(d_{i-1} + d_i)|.$$

Equation (3.6) then follows from (3.4) of Lemma 1. For the final parts rearrange (3.9) to get

$$|(d_{i-1} + d_i)\, \tau_i'' - F_i| \leqslant (e_{i-1} + e_i) \max_j |\tau_j''| \tag{3.10}$$

$$\leqslant 2\rho_{\max}(1/(\rho_i{}^2 h_i) + 1/(\rho_{i-1}^2 h_{i-1})) \max_j |F_j|.$$

But the right-hand side can be made arbitrarily small as $n \to \infty$ so (3.7), (3.8) follow.

Note that (3.7) and (3.8) imply $(\tau^n)_i'' \sim \max(\rho_{i-1}^n, \rho_i{}^n)$ as $n \to \infty$ because of (3.1) and (3.4). Thus the second derivatives at mesh points become unbounded as the tension increases to infinity.

We are now ready to state the main result concerning convexity properties of the tension splines. This has also been given by Schweikert [8] for uniform tension and Spath [9] for variable tension although the latter's argument is incomplete. Both papers are concerned with eliminating extraneous inflection points. Spath essentially defines such an inflection point as a point in (x_i, x_{i+1}) for which τ'' vanishes (equivalently $\tau_i'' \tau_{i+1}'' < 0$) while $F_i F_{i+1} > 0$. Note that if f'' has constant sign in $[a, b]$ then F_i has this same sign for all i, moreover, if f'' has constant sign in $[x_{j-1}, x_{j+1}]$ then F_j has this same sign. Thus $\{F_i\}$ reflects the convexity of f and it is the signs of these quantities which are used to determine the values of the tension parameters.

It is convenient to begin by stating the basic formula (2.1) for the tension spline as well as its first two derivatives. For $x \in [x_i, x_{i+1}]$

$$\tau(x) = [\tau_i'' \sinh \rho_i(x_{i+1} - x) + \tau_{i+1}'' \sinh \rho_i(x - x_i)]/(\rho_i{}^2 S_i)$$
$$+ (f_i - \tau_i''/\rho_i{}^2)(x_{i+1} - x)/h_i + (f_{i+1} - \tau_{i+1}''/\rho_i{}^2)(x - x_i)/h_i, \tag{3.11}$$

$$(D\tau)(x) = [\tau_{i+1}'' \cosh \rho_i(x - x_i) - \tau_i'' \cosh \rho_i(x_{i+1} - x)]/(\rho_i S_i)$$
$$+ (f_{i+1} - f_i)/h_i + (\tau_i'' - \tau_{i+1}'')/(\rho_i{}^2 h_i), \tag{3.12}$$

$$(D^2\tau)(x) = [\tau_{i+1}'' \sinh \rho_i(x - x_i) + \tau_i'' \sinh \rho_i(x_{i+1} - x)]/S_i. \tag{3.13}$$

THEOREM 2. *For a fixed partition, if $\{\tau^n\}$ is a Case 1 sequence of inter-polatory tension splines and F_k, F_{k+1} given by (3.5) are positive (negative) then for n sufficiently large $D^2\tau^n$ is positive (negative) in $[x_k, x_{k+1}]$. If $\{\tau^n\}$ is a Case 2 sequence and F_i is positive (negative) for all i then for n sufficiently large $D^2\tau^n$ is positive (negative) in $[a, b]$.*

Proof. Superscripts are omitted for simplicity. From (3.7) or (3.8) for n sufficiently large τ_k'' (τ_{k+1}'') has the same sign as F_k (F_{k+1}) since $d_i > 0 \, \forall_i$.

But from (3.13) it is clear that if τ_k'' and τ_{k+1}'' have the same sign then $D^2\tau$ has this same sign in $[x_k, x_{k+1}]$.

For the second part (3.10) which holds for each k and n implies

$$|(d_{k-1} + d_k)\tau_k'' - F_k| \leqslant (4\sigma^2/\rho_{\max})(1/h_k + 1/h_{k-1})\max_j |F_j|$$

and as in the previous paragraph for n sufficiently large ($\rho_{\max}$ sufficiently large) τ_k'' has the same sign as F_k, etc.

Spath [9] describes an iterative technique for selecting $\{\rho_i\}$ in order that τ'' has the appropriate sign. His algorithm does not appear to incorporate the formula (2.4) so is subject to loss of significance for small $\rho_i h_i$. Equation (3.10) provides sufficient conditions on ρ_{i-1}, ρ_i so that τ_i'' and F_i have the same sign. In the case where the remaining tension parameters are fixed and smaller it says

$$\rho_{i-1} = \rho_i > (2/|F_i|)(1/h_i + 1/h_{i-1})\max_j |F_j|.$$

This has proven to be very conservative in practice so Spath's approach is more attractive, particularly since it can be used to select all tension parameters so that τ is globally convex (when this is compatible with the data).

Hill [5] uses tension splines to construct monotone approximations to discrete cumulative density functions arising in statistical applications. His proof that for sufficiently large ρ the interpolant is monotone is for the case of uniform tension but can be extended to the variable tension case using a formula of Spath (at least for uniformly bounded tension parameter ratios). Spath claims [9, Eq. 25]

$$\lim_{\rho_{i-1},\rho_i \to \infty} \tau_i' = \tfrac{1}{2}\{f[x_i, x_{i+1}] + f[x_{i-1}, x_i]\} \tag{3.14}$$

where $\tau_i' = \tau'(x_i)$, $f[x_i, x_{i+1}] = (f(x_{i+1}) - f(x_i))/h_i$. Equation (3.14) results from averaging (3.12) with the corresponding formula for the $(i-1)$st interval. Actually one has

$$\tau_i' = \tfrac{1}{2}\{f[x_i, x_{i+1}] + f[x_{i-1}, x_i]\} + \tfrac{1}{2}\{e_{i-1}\tau_{i-1}'' - e_i\tau_{i+1}''\}$$
$$+ \tfrac{1}{2}(d_{i-1} - d_i)\tau_i''.$$

As $\rho_{i-1}, \rho_i \to \infty$ the second term goes to zero since $|\tau_i''| \leqslant 2\rho_{\max}\max_j |F_j|$ from (3.6) and $e_i \leqslant 1/(\rho_i^2 h_i)$ from (3.3). But the third term is $O(1)$ unless $\rho_{\max}/\rho_{\min}$ is uniformly bounded over the sequence. If the latter holds then indeed (3.14) is true. Note that this says for f monotone in $[x_{i-1}, x_{i+1}]$ τ_i' has the same sign as f' for ρ_{i-1}, ρ_i sufficiently large.

At this point it is useful to digress and examine what happens in the interior of $[x_i, x_{i+1}]$ as $\rho_i \to \infty$. Consider $x \in I = [x_i + \delta, x_{i+1} - \delta]$ where $0 < \delta < h_i/2$. Then from (3.13) and (3.6)

$$|(D^2\tau)(x)|$$

$$\leqslant 2\rho_{\max} \max_j |F_j| \{\exp(-\rho_i(x - x_i))(1 - \exp(-2\rho_i(x_{i+1} - x)))$$

$$+ \exp(-\rho_i(x_{i+1} - x))(1 - \exp(-2\rho_i(x - x_i)))\}/(1 - \exp(-2\rho_i h_i))$$

$$\leqslant 4\rho_{\max} \max_j |F_j| \exp(-\rho_i\delta)/(1 - \exp(-2\rho_i h_i)). \tag{3.15}$$

The right-hand side can be made arbitrarily small by taking ρ_i sufficiently large. From the Mean Value Theorem $\exists \xi \in (x_i, x_{i+1})$ such that $\tau(x_{i+1}) = \tau(x_i) + h_i\tau'(\xi)$; this implies $\tau'(x) - f[x_i, x_{i+1}] = \int_\xi^x \tau''(s)\, ds$. Thus if δ is chosen small enough so that $\xi \in I$ then

$$|\tau'(x) - f[x_i, x_{i+1}]| \leqslant h_i \max_I |\tau''|. \tag{3.16}$$

From (3.15) it follows that $\tau' \approx f[x_i, x_{i+1}]$ in I for ρ_i sufficiently large. Finally let $l_i(x) = f_i(x_{i+1} - x)/h_i + f_{i+1}(x - x_i)/h_i$, the linear interpolant for f on $[x_i, x_{i+1}]$. Then from (3.11) for any $x \in [x_i, x_{i+1}]$

$$|\tau(x) - l_i(x)| \leqslant 4 \max(|\tau_i''|/\rho_i^2, |\tau_{i+1}''|/\rho_i^2)$$

$$\leqslant 8\rho_{\max}/\rho_i^2 \max_j |F_j| \tag{3.17}$$

from (3.6). Thus τ converges uniformly to l_i as $\rho_i \to \infty$. These results are summarized in

THEOREM 3.   *If l_i is the linear interpolant for f on $[x_i, x_{i+1}]$ then*

$$\lim_{\rho_i \to \infty} \tau(x) = l_i(x) \qquad uniformly\ in\ [x_i, x_{i+1}].$$

In any closed proper subinterval of $[x_i, x_{i+1}]$

$$\lim_{\rho_i \to \infty} D\tau = Dl_i \qquad uniformly,$$

$$\lim_{\rho_i \to \infty} D^2\tau = 0 \qquad uniformly.$$

Clearly the preceding arguments can be extended to a Case 2 sequence of tension splines where all tension parameters grow but in such a way that $\rho_{\max}/\rho_{\min}$ is uniformly bounded. In this case it can be said that as $\rho_{\max} \to \infty$ τ approaches the linear spline interpolant for f uniformly in $[a, b]$ and the first two derivatives of τ approach those of the linear spline away from the

mesh points. At the mesh points $\tau_i' \to \frac{1}{2}(f[x_{i-1}, x_i] + f[x_i, x_{i+1}])$ and τ_i'' grows in magnitude like ρ_{max}.

If f is monotone on $[a, b]$ then $f[x_i, x_{i+1}]$ has the same sign for all i. Thus the remarks in the previous paragraph lead to the desired result on the monotonicity of τ for ρ_{max} sufficiently large.

THEOREM 4. *For a fixed partition, if $\{\tau^n\}$ is a Case 2 sequence of interpolatory tension splines and $f[x_i, x_{i+1}]$ is positive (negative) for all i then for ρ_{max}^n sufficiently large τ^n is always increasing (decreasing).*

Proof. For ρ_{max}^n sufficiently large τ^n can be made arbitrarily close to the linear spline interpolant away from mesh points. But from (3.14) $(\tau^n)'$ also has the desired sign at mesh points when ρ_{max}^n is sufficiently large. Thus continuity of $(\tau^n)'$ assures that τ^n is monotone.

Hill [5] describes an iterative algorithm for choosing the tension parameters so that τ is monotone for monotone data. The bounds in (3.15), (3.16) also provide sufficient conditions for ρ_i but these are far too conservative in general.

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